

# Pyramids and monomial blowing-ups

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## Abstract

We show that a convex pyramid  $\Gamma$  in  $\mathbb{R}^n$  with apex at  $\mathbf{0}$  can be brought to the first quadrant by a finite sequence of monomial blowing-ups if and only if  $\Gamma \cap (-\mathbb{R}_0^n) = \{\mathbf{0}\}$ . The proof is non-trivially derived from the theorem of Farkas-Minkowski. Then, we apply this theorem to show how the Newton diagrams of the roots of any Weierstraß polynomial

$$P(\mathbf{x}, z) = z^m + h_1(\mathbf{x})z^{m-1} + \cdots + h_{m-1}(\mathbf{x})z + h_m(\mathbf{x}),$$

$h_i(\mathbf{x}) \in k[[x_1, \dots, x_n]][z]$ , are contained in a pyramid of this type. Finally, if  $n = 2$ , this fact is equivalent to the Jung-Abhyankar theorem.

## 1 Introduction

We will operate in the euclidean space  $\mathbb{R}^n$  with its affine structure. As it is classical, we will distinguish between the point-space  $X = \mathbb{R}^n$  and the underlying vector space  $V = \mathbb{R}^n$ . The vector addition is the canonical action (translations) of  $V$  on  $X$ .

A *polyhedron* is the convex hull of a finite set of points generating the affine space  $X$ . Equivalently ([?], page 30), a polyhedron is the compact intersection of a finite set of half-spaces. Moreover, if  $E = \{A_0, A_1, \dots, A_m\}$  is a finite set of points generating  $X$  and, if  $\mathcal{H} = \{H_1, \dots, H_p\}$  is the set of all (different) hyperplanes passing through all possible subsets of  $E$  consisting of  $n$  affinely independent points, then the set of vertices of the convex hull  $[E]$  of  $E$  is the set of points of intersection of all the subsets of  $\mathcal{H}$  consisting of  $n$  hyperplanes whose intersection is an only point<sup>1</sup>.

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<sup>1</sup>c.f., for instance, Vicente, J.L.: *Notas sobre convexidad* at <http://www.us.es/da>

A *pyramid* is the projection, from the origin  $\mathbf{0} \in \mathbb{R}^n$ , of a polyhedron contained in a hyperplane  $H$  not passing through  $\mathbf{0}$ . To be more precise: given a hyperplane  $H$  such that  $\mathbf{0} \notin H$  and a finite set  $E$  of points generating  $H$ , denoting by  $\Delta$  the convex hull  $[E]$  of  $E$ , then the corresponding pyramid is

$$\Gamma(\Delta) = \bigcup_{\mathbf{a} \in \Delta} \langle \mathbf{a} \rangle_+,$$

where  $\langle \mathbf{a} \rangle_+$  is the half-line of the non-negative multiples of  $\mathbf{a}$ . Equivalently (c.f. Vicente, J.L., loc.cit.), a pyramid can be given by a polyhedron  $\Delta'$  in  $X$ , one of whose vertices is  $\mathbf{0}$ . In this case, the pyramid is nothing but

$$\Gamma = \bigcup_{\mathbf{a} \in \Delta' \setminus \{\mathbf{0}\}} \langle \mathbf{a} \rangle_+.$$

Moreover, if  $\{\mathbf{0}, \mathbf{a}_1, \dots, \mathbf{a}_m\}$  are the vertices of  $\Delta'$ , there is a hyperplane  $H$  strictly separating  $\mathbf{0}$  from  $[\mathbf{a}_1, \dots, \mathbf{a}_m]$ . Then  $\Delta = H \cap \Delta'$  is a polyhedron in  $H$  and  $\Gamma = \Gamma(\Delta)$ .

**Definition 1.1.**— *We will call a monomial blowing-up (resp. a monomial blowing-down) any linear automorphism of  $\mathbb{R}^n$  of the form*

$$(a_1, \dots, a_n) \rightarrow (a_1, \dots, a_n)M_{ij} \quad (\text{resp.} \quad (a_1, \dots, a_n) \rightarrow (a_1, \dots, a_n)N_{ij})$$

where:

1.  $i, j \in \mathbb{Z}, i \neq j, 1 \leq i, j \leq n$
2.  $M_{ij}$  (resp.  $N_{ij}$ ) is equal to the identity matrix in which the  $(i, j)$ -entry is set to 1 (resp. to  $-1$ ).

**Remark 1.2.**— With the notations of definition 1.1, the monomial blowing-up (resp. monomial blowing-down) acts in the following way:

$$\begin{aligned} (a_1, \dots, a_n) &\rightarrow (a_1, \dots, a_i \overset{j)}{+} a_j, \dots, a_n) \\ \text{resp.} \quad (a_1, \dots, a_n) &\rightarrow (a_1, \dots, a_j \overset{j)}{-} a_i, \dots, a_n) \end{aligned}$$

This corresponds to the behavior of the exponents of a monomial under the geometric monomial blowing-up  $x_i \rightarrow x_i x_j$  or the monomial blowing-down  $x_i \rightarrow x_i / x_j$ . In fact, this geometric monomial blowing-up (resp. monomial blowing-down) acts on monomials in the following way:

$$\begin{aligned} x_1^{a_1} \cdots x_j^{a_j} \cdots x_n^{a_n} &\rightarrow x_1^{a_1} \cdots x_j^{a_i + a_j} \cdots x_n^{a_n} \\ \text{resp.} \quad x_1^{a_1} \cdots x_j^{a_j} \cdots x_n^{a_n} &\rightarrow x_1^{a_1} \cdots x_j^{a_j - a_i} \cdots x_n^{a_n}. \end{aligned}$$

This is the reason of the name for these linear automorphisms.

From now on, we will use the name of monomial blowing-up (resp. monomial blowing-down) indistinctly for the linear automorphisms defined in 1.1 or for the polynomial substitutions  $x_i \rightarrow x_i x_j$  (resp.  $x_i \rightarrow x_i/x_j$ ).

**Remark 1.3.**— Let  $E = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$  be a finite set of points generating a hyperplane  $H$  which does not contain the origin, let  $\Delta = [E] \subset H$  be the corresponding polyhedron and  $\Gamma(\Delta)$  the pyramid; then

$$\Gamma(\Delta) = \left\{ \sum_{i=1}^m \lambda_i \mathbf{a}_i \mid \lambda_i \geq 0, \forall i = 1, \dots, m \right\}.$$

Equivalently, let  $A$  be the matrix whose row vectors are  $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ ; then

$$\Gamma(\Delta) = \{(\lambda_1, \dots, \lambda_m)A \mid (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_0^m\}.$$

If  $\varphi_{ij}$  is the monomial blowing-up (resp. the monomial blowing-down) with matrix  $M_{ij}$  (resp.  $N_{ij}$ ) and  $\mathbf{a}'_i = \varphi_{ij}(\mathbf{a}_i)$  then  $E' = \{\mathbf{a}'_1, \dots, \mathbf{a}'_m\}$  generates a hyperplane  $H'$  not passing through  $\mathbf{0}$ . If  $\Delta' = [E']$ , then  $\Gamma(\Delta') = \varphi(\Gamma(\Delta))$ , so it makes sense to speak on the transform of a pyramid by a monomial blowing-up or a monomial blowing-down.

**Definition 1.4.**— The first quadrant of  $X$  is the set  $\mathbb{R}_0^n$ . The opposite of the first quadrant of  $X$  is the set  $-\mathbb{R}_0^n$ .

The main problem we deal with in this paper is whether, given a pyramid  $\Gamma(\Delta)$ , it exists a finite sequence of monomial blowing-ups such that the transform of the pyramid by the sequence is contained in the first quadrant. We solve it by giving a geometrical criterion, from which we derive explicit computations using existing optimization algorithms. The criterion is the following:

**Theorem 1.5.**— Let  $E = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$  be a finite set of points generating a hyperplane  $H$  not containing the origin, let  $\Delta = [E]$  be the corresponding polyhedron and  $\Gamma(\Delta)$  the pyramid; then the following conditions are equivalent:

1. There exists a finite sequence of monomial blowing-ups such that the transform of  $\Gamma(\Delta)$  by the sequence is contained in the first quadrant.
2.  $\Gamma(\Delta) \cap (-\mathbb{R}_0^n) = \{\mathbf{0}\}$ .

The second condition can be easily checked by the simplex method; in remark 4.2 we will show how. Let us observe that the first condition implies the second because  $-\mathbb{R}_0^n$  is stable by monomial blowing-ups. In

fact, if  $\Gamma(\Delta)$  had a point  $\mathbf{a} \neq \mathbf{0}$  in common with  $-\mathbb{R}_0^n$  then, no matter what sequence of monomial blowing-ups we apply, the transform of  $\mathbf{a}$  will stay in  $-\mathbb{R}_0^n$ . The point is then to prove that the second condition implies the first.

As an application, we show that the Newton diagrams of all the roots of a Weierstraß polynomial

$$P(\mathbf{x}, z) = z^m + h_1(\mathbf{x})z^{m-1} + \cdots + h_{m-1}(\mathbf{x})z + h_m(\mathbf{x}) \in k[[\mathbf{x}]] [z], \quad m > 1$$

are contained in a pyramid satisfying the equivalent conditions of theorem 1.5. Moreover, we show how, in dimension 2, this fact is in some sense equivalent to the Jung-Abhyankar theorem (c.f. [?]).

## 2 The proof

**Remark 2.1.**— Let us denote by  $A$  the matrix whose row vectors are  $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ . We will speak of “bringing  $A$  to the first quadrant” as equivalent to bringing  $\Gamma(\Delta)$  to the first quadrant. It is obvious that, if  $A$  has column with only positive entries, then it can be brought to the first quadrant by a finite sequence of monomial blowing-ups: it is enough to add this column to the others a suitable number of times.

We use linear optimization methods to prove theorem 1.5. We refer to [?] for the theorem of Farkas-Minkowski and its consequences. In particular, we take from it (pages 42-51) the following consequence of this theorem (which might be also taken as an alternative statement of it):

**Corollary 2.2.**— *Let  $A$  be a matrix  $m \times n$ ; then one and only one of the following conditions hold:*

1. *There exists a non-zero vector  $x \geq 0$  such that  $xA \leq 0$ .*
2. *The system of inequalities  $Ay > 0$  has a non-negative solution.*

*Inequalities must be understood componentwise.*

From this result we derive the following, which is the useful one:

**Corollary 2.3.**— *Let  $A$  be a matrix  $m \times n$ ; then one and only one of the following conditions holds:*

1. *There exists a non-zero vector  $x \geq 0$  such that  $xA \leq 0$ .*
2. *The system of inequalities  $Ay > 0$  has a positive integer solution (that is, a vector of positive integers).*

**Proof:** By corollary 2.2, we must only prove that the existence of a non-negative real solution of  $Ay > 0$  implies that there is a positive integer one. Hence, it is enough to show the existence of a positive rational solution.

Let  $y$  be a non-negative real column vector such that  $Ay > 0$ , let  $a_i$  be the rows of  $A$ ,  $i = 1, \dots, m$ , and let  $a_i y = \alpha_i > 0$ . Let  $0 < \varepsilon < \min\{\alpha_i\}$  and let  $K$  be a polidisc centered at  $y$  such that, for all  $z \in K$ , one has  $|\alpha_i - a_i z| < \varepsilon$ . Then,  $-\varepsilon < a_i z - \alpha_i < \varepsilon$ , so  $\alpha_i - \varepsilon < a_i z < \varepsilon + \alpha_i$ , which implies that  $a_i z > 0$ . Since  $K$  contains rational vectors greater than zero, the corollary is proven.  $\square$

**Proposition 2.4.**— *Let  $n > 1$  and  $y = (y_1, \dots, y_n)$  be a vector of positive integers with greatest common divisor equal to 1. There exists a  $n \times n$  matrix  $Y$ , whose entries are non-negative integers, with determinant equal to 1, one of whose columns is  $y$ .*

**Proof:** Let  $a \in \mathbb{Z}$  and denote by  $E_{ij}(a)$ ,  $i \neq j$ , the  $n \times n$  elementary matrix, which is equal to the identity matrix with its  $(i, j)$ -entry replaced by  $a$ . Let us write  $y$  as a column vector.

Let us assume that all the  $y_j$  are multiple of one of them, say  $y_i$ ; then it must be  $y_i = 1$ . Left multiplications by matrices  $E_{ji}(-y_j)$  allow us to transform the column vector  $y$  into a column vector having all entries equal to zero, except the  $i$ -th one which is equal to 1. Remark that all the elementary matrices we have used have a *negative* entry out of the main diagonal.

Let us assume that no entry of the column vector  $y$  divides all the others and let  $y_i$  be the smallest of all these entries. By assumption, there must be a  $j$  such that, in the euclidean division,  $y_j = q_j y_i + r_j$  with  $0 < r_j < y_i$ . Left multiplication of  $y$  by  $E_{ji}(-q_j)$  puts  $r_j$  at the position  $j$ , leaving the other entries unchanged. In this way, we get a new column vector such that the greatest common divisor of its entries is 1 and the minimum of these entries has strictly decreased. Remark that, again, we have used an elementary matrix with a negative integer entry out of the main diagonal. If we repeat this process, after a finite number of steps, we fall in the preceeding situation.

Summing-up: we have proven that, by left multiplication of the column vector  $y$  by elementary matrices having negative integer entries out of the main diagonal, we arrive at a matrix which is a column of the identity matrix. If we denote by  $Y$  the inverse of this product of elementary matrices, we see that  $Y$  is a product of elementary matrices with positive integer entries out of the main diagonal, so  $Y$  is a matrix with non-negative integer entries. Let  $I_n$  be the  $n \times n$  identity matrix; then it is clear that  $Y = Y I_n$  contains a column equal to  $y$  and, of course,  $\det(Y) = 1$ . This proves the proposition.  $\square$

**Proposition 2.5.**— *Let  $Y$  be a square matrix with non-negative integer entries whose determinant is equal to 1. Then  $Y$  can be written as a product of (by order): a finite number of monomial blowing-up matrices, a permutation matrix and another finite number of monomial blowing-up matrices.*

**Proof:** It is enough to show that  $Y$  can be brought to a permutation matrix by left and right multiplication by monomial blowing-down matrices. If  $q \in \mathbb{Z}_+$ , then  $E_{ij}(-q) = E_{ij}(-1)^q$ ; since  $E_{ij}(-1)$  is a monomial blowing-down matrix, then  $E_{ij}(-q)$  is a product of monomial blowing-down matrices.

Let  $y_{ij}$  be the smallest non-zero entry in  $Y$ . If all the elements in the  $j$ -th column are multiple of  $y_{ij}$ , we can get zeros in all the positions of this column, except  $(i, j)$ , by left multiplication by elementary matrices with a negative entry out of the main diagonal. In this case, the fact that  $\det(Y) = 1$  implies  $y_{ij} = 1$ . If some entry in the  $j$ -th column is not a multiple of  $y_{ij}$ , say  $y_{lj}$ , and if  $y_{lj} = qy_{ij} + r_{lj}$  is the euclidean division, then left multiplication by  $E_{jl}(-q)$  puts  $r_{lj}$  at the position  $(l, j)$  so the smallest non-zero entry of  $Y$  has strictly decreased. If we repeat this process, it is clear that we must arrive to the first case after a finite number of steps. The end of this process is a matrix  $Y'_1$  with a 1 at one position (denote it again by  $(i, j)$ ) and zeros everywhere else in the  $j$ -th column. Moreover,  $Y'_1$  is the result of left multiplying  $Y$  by monomial blowing-down matrices.

By symmetry, it is clear that we can get a new matrix  $Y_1$ , obtained from  $Y'_1$  by right multiplication by monomial blowing-down matrices, and having 1 at the  $(i, j)$  position and zeros everywhere else in the  $i$ -th row and the  $j$ -th column. This is the basic argument of our proof.

We may repeat the argument for the submatrix of  $Y_1$  obtained by deleting the  $i$ -th row and the  $j$ -th column, but seeing the operations in the whole  $Y_1$ . This does not affect the form of  $Y_1$ . The very end of the process is a matrix  $Y_p$ , which is a permutation of the rows of the identity matrix, and which is obtained from  $Y$  by left and right multiplication by monomial blowing-down matrices. This proves the proposition.  $\square$

**Lemma 2.6.**— *In the situation of theorem 1.5, let  $A$  be the matrix whose row vectors are  $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ . For all  $x = (x_1, \dots, x_m) \in \mathbb{R}_0^m \setminus \{\mathbf{0}\}$  one has*

$$0 \neq \sum_{i=1}^m x_i \mathbf{a}_i.$$

**Proof:** Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be a linear function such that  $f(\mathbf{a}_i) > 0, \forall i = 1, \dots, m$ . There always exists such a function: we give an example.

Let  $g$  be an affine function such that  $H$  has the equation  $g = 0$  and  $g(\mathbf{0}) < 0$ . Then, for all  $i = 1, \dots, m$  one has  $0 = g(\mathbf{a}_i) = g(\mathbf{0}) + \vec{g}(\mathbf{a}_i)$ , so  $\vec{g}(\mathbf{a}_i) > 0$  and we may take  $f = \vec{g}$ .

Then,

$$f\left(\sum_{i=1}^m x_i \mathbf{a}_i\right) = \sum_{i=1}^m x_i f(\mathbf{a}_i) > 0,$$

which proves the lemma.  $\square$

**Remark 2.7.**— PROOF OF THEOREM 1.5:

Let us assume that  $\mathbf{0}$  is the only point of  $\Gamma(\Delta)$  belonging to  $-\mathbb{R}_0^n$ . By lemma 2.6, for every  $x \in \mathbb{R}_0^m \setminus \{\mathbf{0}\}$ , the vector  $xA$  is different from zero. By assumption, it cannot be  $xA \leq 0$ . By corollary 2.3, there must exist a vector  $y \in \mathbb{Z}_+^n$  such that, written as a column vector,  $Ay > 0$ . We may assume that the greatest common divisor of the entries of  $y$  is 1. By proposition 2.4, there exists a matrix  $Y$  with non-negative integer entries and determinant equal to 1 such that  $y$  is one of its columns. Therefore, one of the columns of  $AY$  is  $Ay > 0$ , which implies by remark 2.1 that  $AY$  can be brought to the first quadrant by a finite sequence of monomial blowing-ups. By proposition 2.5, we obtain  $AY$  from  $A$  by applying to  $A$  a finite sequence of monomial blowing-ups, then a permutation of the columns and then another finite sequence of monomial blowing-ups. It is evident that the permutation of the columns plays no role: if the matrix with the permuted columns can be brought to the first quadrant, also the original one. This proves the theorem.

### 3 Applications

The main application of theorem 1.5 we consider here lies in the resolution of equations of the form: a Weierstraß polynomial equal to zero. Let  $k$  be an algebraically closed field of characteristic zero,  $\mathbf{x} = (x_1, \dots, x_n)$  a collection of indeterminates,  $R = k[[\mathbf{x}]]$  the corresponding ring of power series, and let

$$P(\mathbf{x}, z) = z^m + h_1(\mathbf{x})z^{m-1} + \dots + h_{m-1}(\mathbf{x})z + h_m(\mathbf{x}) \in k[[\mathbf{x}]][[z]], \quad n > 1$$

be an irreducible Weierstraß polynomial; the object to study is the equation  $P(\mathbf{x}, z) = 0$ . Let  $D \in k[[\mathbf{x}]]$  be the discriminant of  $P$  with respect to  $z$ ; the Jung-Abhyankar theorem (c.f. [?]) asserts that, if  $D$  is of the form  $\mathbf{x}^{\mathbf{a}}U(\mathbf{x})$  with  $U(\mathbf{x}) \in k[[\mathbf{x}]]$ ,  $U(\mathbf{0}) \neq 0$ , then the roots of  $P(\mathbf{x}, z) = 0$  are a full set of conjugate Puiseux power series in the variables  $\mathbf{x}$ . Here,  $\mathbf{x}^{\mathbf{a}}$  means  $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ , where  $\mathbf{a} = (a_1, \dots, a_n) \in$

$\mathbb{Z}_0^n \setminus \{\mathbf{0}\}$  When the discriminant has this very special form, we say that it is a *normal crossing divisor*.

In general, the roots are not Puiseux power series in  $\mathbf{x}$ . However, we can say something very important about them, namely

**Theorem 3.1.**— *The roots of  $P(\mathbf{x}, z) = 0$  are power series belonging to a ring  $k((x_n^{1/p})) \cdots ((x_{i+1}^{1/p}))[[x_1^{1/p}, \dots, x_i^{1/p}]]$  whose Newton diagrams are contained in a pyramid  $\Gamma(\Delta)$  such that  $\Gamma(\Delta) \cap (-\mathbb{R}_0^n) = \{\mathbf{0}\}$ .*

We will prove the theorem through several remarks.

**Remark 3.2.**— We take the lexicographic order in the sense (c.f. [?], page 50): if  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$  then  $\mathbf{a} <_{\text{lex}} \mathbf{b}$  if and only if the first component (from left to right) of  $\mathbf{a}$  which is different of the corresponding in  $\mathbf{b}$  is strictly smaller.

Let us observe that a monomial blowing-up  $\varphi_{ij}$  of the type  $\mathbf{a} \rightarrow \mathbf{a}M_{ij}$  with  $i < j$  preserves the lexicographic order, so it is an ordered automorphism of  $\mathbb{R}^n$  (endowed with the lexicographic order). In fact,

$$\varphi_{ij}(a_1, \dots, a_i, \dots, a_j, \dots, a_n) = (a_1, \dots, a_i, \dots, a_i + a_j, \dots, a_n),$$

so, if  $\mathbf{a} <_{\text{lex}} \mathbf{b}$ , then:

1. If  $<_{\text{lex}}$  is decided before the position  $j$ , it is evident that  $\varphi(\mathbf{a}) <_{\text{lex}} \varphi(\mathbf{b})$ .
2. If  $<_{\text{lex}}$  is decided at the position  $j$ , this means that  $a_l = b_l, \forall l, 1 \leq l \leq j-1$  and  $a_j < b_j$ . Therefore,  $a_i + a_j < b_i + b_j$ , hence  $\varphi(\mathbf{a}) <_{\text{lex}} \varphi(\mathbf{b})$ .
3. If  $<_{\text{lex}}$  is decided at a position  $l$  after  $j$ , this means that all the components of  $\mathbf{a}$  until the  $(l-1)$ -th coincide with the corresponding in  $\mathbf{b}$ , so the same happens with  $\varphi(\mathbf{a})$  and  $\varphi(\mathbf{b})$ . Since  $a_l < b_l$  then  $\varphi(\mathbf{a}) <_{\text{lex}} \varphi(\mathbf{b})$ .

We call this an *order-preserving monomial blowing-up*. Notice that the corresponding monomial blowing-down  $\varphi_{ij}^{-1}$  is also order-preserving.

**Remarks 3.3.**— Let  $\Lambda \subset \mathbb{Z}_0^n$  be a non-empty cloud of points; we call the transform of  $\Lambda$  by a monomial blowing-up (or a monomial blowing-down) the set of the transforms of all the points of  $\Lambda$ .

3.3.1. Any monomial blowing-up  $\varphi$  keeps  $\mathbb{Z}_0^n$ , that is,  $\varphi(\mathbb{Z}_0^n) \subset \mathbb{Z}_0^n$ . Therefore, if  $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_0^n$  are two points such that  $\mathbf{b} \in \mathbf{a} + \mathbb{Z}_0^n$  then  $\varphi(\mathbf{b}) \in \varphi(\mathbf{a}) + \mathbb{Z}_0^n$ .

3.3.2. Let us assume that  $\varphi = \varphi_{pq}, p < q$ , is an order-preserving monomial blowing-up. Let  $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_0^n$  and write  $\mathbf{a}' = \varphi(\mathbf{a}), \mathbf{b}' = \varphi(\mathbf{b})$ . Let us assume that there exists an index  $j, 1 \leq j \leq n$  such that



$b_i \geq a_i, \forall i = 1, \dots, j$ ; then  $b'_i \geq a'_i, \forall i = 1, \dots, j$ . In fact, the result is clear by 3.3.1 if  $j = n$ , so let us assume that  $j < n$ . Let  $\mathbf{c} = (b_1 - a_1, \dots, b_j - a_j, 0, \dots, 0)$  and  $\mathbf{d} = (0, \dots, 0, b_{j+1} - a_{j+1}, \dots, b_n - a_n)$ ; then  $\mathbf{b} = \mathbf{a} + \mathbf{c} + \mathbf{d}$ . Since  $\mathbf{a} + \mathbf{c} \in \mathbf{a} + \mathbb{Z}_0^n$ , then  $\varphi(\mathbf{a} + \mathbf{c}) \in \varphi(\mathbf{a}) + \mathbb{Z}_0^n$ . On the other hand, since  $p < q$ , the first  $j$  components of  $\varphi(\mathbf{d})$  are zero, so the conclusion is clear. It is obvious that the same happens if we replace  $\varphi_{pq}$  by the composition of a finite sequence of order-preserving monomial blowing-ups.

3.3.3. For any  $j = 1, \dots, n$  and any  $\mathbf{v} \in \mathbb{Z}_0^n$ , we write  $\Lambda_j(\mathbf{v}) = \{\mathbf{v}' \in \mathbb{Z}_0^n \mid v'_i \geq v_i, \forall i = 1, \dots, j\}$ ; then  $\Lambda_1(\mathbf{v}) \supset \dots \supset \Lambda_n(\mathbf{v})$ . By 3.3.2, for every order-preserving monomial blowing-up  $\varphi$ , every  $\mathbf{v} \in \mathbb{Z}_0^n$  and every  $j = 1, \dots, n$ , one has  $\varphi(\Lambda_j(\mathbf{v})) \subset \Lambda_j(\varphi(\mathbf{v}))$ . Moreover, if  $\Phi$  is a composition of a finite number of order-preserving monomial blowing-ups, then  $\Phi(\Lambda_j(\mathbf{v})) \subset \Lambda_j(\Phi(\mathbf{v}))$ .

3.3.4. Since  $\emptyset \neq \Lambda \subset \mathbb{Z}_0^n$ , there is a minimum-lex  $\mathbf{u} \in \Lambda$ , so  $\Lambda \subset \Lambda_1(\mathbf{u})$ . We will now prove that there exists a finite sequence of order-preserving monomial blowing-ups such that, calling  $\Phi$  the composition of all of them, one has  $\Phi(\Lambda) \subset \Phi(\mathbf{u}) + \mathbb{Z}_0^n$ . If  $\Lambda \subset \Lambda_n(\mathbf{u})$  there is nothing to prove, so we assume this is not the case. Let  $j$  be the smallest index such that  $\Lambda \not\subset \Lambda_j(\mathbf{u})$ ; then, necessarily  $j > 1$ . By the minimality of  $j$ , for every  $\mathbf{u}' \in \Lambda$  one must have  $u'_i \geq u_i, \forall i = 1, \dots, j - 1$ . Moreover, if  $\mathbf{u}' \in \Lambda \setminus \Lambda_j(\mathbf{u})$  then  $u'_j < u_j$ , so  $u_j > 0$ . Since  $\mathbf{u} \in \Lambda$  is the minimum-lex, for every  $\mathbf{u}' \in \Lambda \setminus \Lambda_j(\mathbf{u})$  there must exist an index  $i < j$  such that  $u'_i > u_i$ . Let  $i_1 < j$  be the smallest index such that there exists  $\mathbf{u}' \in \Lambda \setminus \Lambda_j(\mathbf{u})$  satisfying  $u'_{i_1} > u_{i_1}$ . Let  $\Phi_1$  be the composition of  $u_j$  monomial blowing-ups equal to  $\varphi_{i_1 j}$ ; then  $\Phi_1(\mathbf{u}) = (u_1, \dots, u_{i_1}, \dots, u_j u_{i_1} + u_j, \dots, u_n)$  and, for every  $\mathbf{u}' \in \Lambda \setminus \Lambda_j(\mathbf{u})$  with  $u'_i > u_i$ ,  $\Phi_1(\mathbf{u}') = (u'_1, \dots, u'_{i_1}, \dots, u_j u'_{i_1} + u'_j, \dots, u'_n)$ . Since  $u'_{i_1} > u_{i_1}$ , then  $u_j(u'_{i_1} - u_{i_1}) \geq u_j \geq u_j - u'_j$ , hence  $u_j u'_{i_1} + u'_j \geq u_j u_{i_1} + u_j$ , so  $\Phi_1(\mathbf{u}') \in \Lambda_j(\Phi_1(\mathbf{u}))$ .

Now,  $\Phi_1(\mathbf{u})$  is the minimum-lex of  $\Phi_1(\Lambda)$  and, by 3.3.3,  $\Phi_1(\Lambda_i(\mathbf{u})) = \Lambda_i(\Phi_1(\mathbf{u})), \forall i = 1, \dots, n$ . By 3.3.2,  $\Phi_1(\Lambda) \subset \Lambda_i(\Phi_1(\mathbf{u})), \forall i = 1, \dots, j - 1$ . One must not forget that  $\Phi_1$  leaves invariant the first  $j - 1$  components of every vector. Therefore, if  $\Phi_1(\Lambda) \not\subset \Lambda_j(\Phi_1(\mathbf{u}))$ , there exists a smallest index  $i_2$  such that there exists  $\mathbf{u}' \in \Phi_1(\Lambda) \setminus \Lambda_j(\Phi_1(\mathbf{u}))$  satisfying  $u'_{i_2} > u_{i_2}$ . Necessarily  $i_2 > i_1$  and we proceed as before, and so on. It is then clear that there exists a finite sequence of order-preserving monomial blowing-ups, whose composition  $\bar{\Phi}$  is such that  $\bar{\Phi}(\Lambda) \subset \Lambda_i(\bar{\Phi}(\mathbf{u})), \forall i = 1, \dots, j$ . If there is still a  $j_1$  such that  $\bar{\Phi}(\Lambda) \not\subset \Lambda_{j_1}(\bar{\Phi}(\mathbf{u}))$ , then  $j_1 > j$  and we proceed as before, and so on. This proves our assertion.

**Remark 3.4.**— PROOF OF THEOREM 3.1. Let  $\Lambda$  be the Newton

diagram of the discriminant  $D$  of  $P(\mathbf{x}, z)$ ; by 3.3.4 there exists a finite sequence of order-preserving monomial blowing-ups such that, calling  $\Phi$  their composition,  $\Phi(\Delta) \subset \mathbf{a} + \mathbb{Z}_0^n$  where  $\mathbf{a} \in \Phi(\Delta)$ . We make these monomial blowing-ups to act upon  $P(\mathbf{x}, z)$  and denote by  $Q(\mathbf{x}, z)$  the transform of  $P(\mathbf{x}, z)$  by  $\Phi$ . The discriminant  $D'$  of  $Q(\mathbf{x}, z)$  is just the transform of  $D$  because  $D$  is a polynomial in the coefficients of the equation. Moreover,  $D'$  is a normal crossing divisor, hence the roots of  $Q = 0$  are all ordinary Puiseux power series, say with common denominator  $p$  of the exponents, because every irreducible factor of  $Q(\mathbf{x}, z)$  has a discriminant which is a normal crossing divisor. If we come back to the beginning by applying the corresponding sequence of monomial blowing-downs, the region containing the Newton diagram of the roots of  $Q = 0$ , namely the first quadrant, obviously goes to a pyramid  $\Gamma(\Delta)$  such that  $\Gamma(\Delta) \cap (-\mathbb{R}_0^n) = \{\mathbf{0}\}$ . Since all the monomial blowing-ups are of the form  $\varphi_{lj}$  with  $l < j$ , we denote by  $i$  the minimum of all the indices  $l$  of these monomial blowing-ups, then  $\Phi$  leaves invariant the first  $i$  coordinates of every point, so the same happens with  $\Phi^{-1}$ . Therefore, every monomial  $x_1^{a_1/p} \cdots x_i^{a_i/p} x_{i+1}^{a_{i+1}/p} \cdots x_n^{a_n/p}$  occurring in a root of  $Q$  evolves in a way such that the exponents  $a_1/p, \dots, a_i/p$  remain unchanged. Therefore, if we fix a root  $\varrho$  of  $Q = 0$ , fix  $a_1/p, \dots, a_i/p$  and write  $\varrho' = x_1^{a_1/p} \cdots x_i^{a_i/p} \varrho''(x_{i+1}^{a_{i+1}/p}, \dots, x_n^{a_n/p})$ , with  $\varrho''(x_{i+1}^{a_{i+1}/p}, \dots, x_n^{a_n/p}) \in k[[x_{i+1}^{1/p}, \dots, x_n^{1/p}]]$ , for the sum of all the terms of the root whose monomials start by  $x_1^{a_1/p} \cdots x_i^{a_i/p}$ , the transform of  $\varrho'$  by  $\Phi^{-1}$  produces a power series  $x_1^{a_1/p} \cdots x_i^{a_i/p} \varrho_1''(x_{i+1}^{a_{i+1}/p}, \dots, x_n^{a_n/p})$  where  $\varrho_1''$  has possibly negative exponents. Since  $(1/p) \cdot \mathbb{Z}_0^n$  is lexicographically well-ordered, so it is  $\Phi^{-1}((1/p) \cdot \mathbb{Z}_0^n)$ , hence the transform of  $\varrho$  by  $\Phi^{-1}$  belongs to  $k((x_n^{1/p})) \cdots ((x_{i+1}^{1/p}))[[x_1^{1/p}, \dots, x_i^{1/p}]]$ , which proves the theorem.

When  $n = 2$  there is much more to say, namely:

**Remark 3.5.**— In our joint paper (cf. [?]), we prove the following for  $n = 2$ :

1. The theorem 3.1 *without using the Jung-Abhyankar theorem*.
2. The Jung-Abhyankar theorem from the fact that the Newton diagrams of the roots lie in a pyramid  $\Gamma(\Delta)$  such that  $\Gamma(\Delta) \cap (-\mathbb{R}_0^2) = \{\mathbf{0}\}$ .

This shows that the Jung-Abhyankar theorem in dimension 2 can be proven by linear algebra techniques, without having resource to more sophisticated algebraic material. Moreover, in this case, the Jung-Abhyankar theorem is equivalent to the fact that the roots of the equation lie in a pyramid satisfying the conditions of theorem 1.5.

## 4 Short remarks on computations

The explicit computations are a consequence, more or less obvious, of the convex calculus and the optimization of a linear function on a polyhedron by the simplex method.

The point of departure will be always the list of points  $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ , all different from  $\mathbf{0}$ , generating, either a hyperplane not passing through the origin, or  $X = \mathbb{R}^n$ . We add  $\mathbf{0}$  to the list, and write  $E = \{\mathbf{0}, \mathbf{a}_1, \dots, \mathbf{a}_m\}$ ; in both cases  $E$  generates the whole affine space. We denote by  $A$  the matrix whose row vectors are  $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ .

**Remark 4.1.**— By elementary linear calculus (c.f. Vicente, J.L., loc. cit.), the  $(n-1)$ -dimensional faces of the polyhedron  $[E]$  are produced by the following algorithm: we pick all the subsets of  $E$  consisting of  $n$  affinely independent points, and find the hyperplane determined by them; then we drop repetitions and keep only those hyperplanes leaving all the points of  $E$  in an only half-space. This algorithm is not the best possible, but improvements are out of the scope of this paper. If  $C = \{H_1, \dots, H_p\}$  is the list of faces, then we get the vertices of  $[E]$  by the following algorithm: we pick all the subsets of  $C$  consisting of  $n$  hyperplanes whose intersection is an only point, find the point, drop repetitions and the remaining ones are the vertices. It is clear that  $E$  defines a pyramid  $\Gamma$  if and only if  $\mathbf{0}$  is a vertex of  $[E]$ . In this case, the faces of  $\Gamma$  are those  $H_i$  passing through  $\mathbf{0}$ . For instance, if we start from the points

$$\mathbf{a}_1 = (4, 7, -9), \mathbf{a}_2 = (5, 7, -8), \mathbf{a}_3 = (3, 5, -9), \mathbf{a}_4 = (4, 0, -1),$$

which generate  $\mathbb{R}_0^n$ , the faces are

$$\begin{array}{ll} 7x_1 - 13x_2 - 7x_3 = 0 & 18x_1 - 9x_2 + x_3 = 0 \\ -7x_1 - 27x_2 - 28x_3 = 0 & 5x_1 + 33x_2 + 20x_3 = 0 \\ 19 - 2x_1 + x_2 + 2x_3 = 0 & 32 - 7x_1 + 5x_2 + 4x_3 = 0 \end{array}$$

and the vertices are  $E = \{\mathbf{0}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ ; therefore the points define a pyramid. The faces are normalized in the sense that all the points of  $E$  make their linear equations  $\geq 0$ . The faces of the pyramid are the first four and the edges are the positive half-lines determined by the four given points.

**Remark 4.2.**— It is not difficult to know whether the pyramid  $\Gamma$  satisfies the condition  $\Gamma \cap (-\mathbb{R}_0^n) = \{\mathbf{0}\}$  or not. Let  $\Lambda = (\lambda_1, \dots, \lambda_m)$  be a row of variables and let  $m_i$  be the element in the  $i$ -th column of the matrix  $\Lambda A$ ; then  $\Gamma \cap (-\mathbb{R}_0^n) \neq \{\mathbf{0}\}$  if and only if there is a feasible

solution to the set of linear constraints

$$1 + \sum_{i=1}^m m_i = 0, \quad m_i \leq 0, \quad \lambda_i \geq 0, \quad i = 1, \dots, m.$$

The existence of a feasible solution can obviously be decided by the simplex method. In the preceding example, the feasible solution does not exist, so  $\Gamma \cap (-\mathbb{R}_0^n) = \{\mathbf{0}\}$ .

**Remark 4.3.**— It is also easy to find a positive solution of the system of inequalities  $Ay > 0$ , where  $y$  is a column of variables. If  $m_i$  is the element in the  $i$ -th row of  $Ay$ , we can easily get a positive solution of  $Ay > 0$  by minimizing any of the coordinate functions on the set of constraints  $m_i \geq 1, y_i \geq 1, i = 1, \dots, m$ . In our example, minimizing  $y_1$  by the simplex method will produce the point  $(1, 7/5, 1)$ , so the integer solution  $(5, 7, 5)$ .

The remaining computations to bring  $A$  to the first quadrant are straightforward matrix operations.